Quantum ergodicity in the SYK model

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Nucl. Phys. B **930**, 45-68 (2018)

Minneapolis, 4. May. 2018

Sachdev-Ye-Kitaev Model (15)

A model of *N* randomly interacting *Majorana* fermions

$$\boxed{\hat{H} = \sum_{ijkl}^{N} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l, \qquad \{\chi_i, \chi_j\} = 2\delta_{ij}}$$

SYK model

where the interaction constants are static and random,

$$\langle |J_{ijkl}|^2 \rangle = \frac{6J^2}{N^3}$$

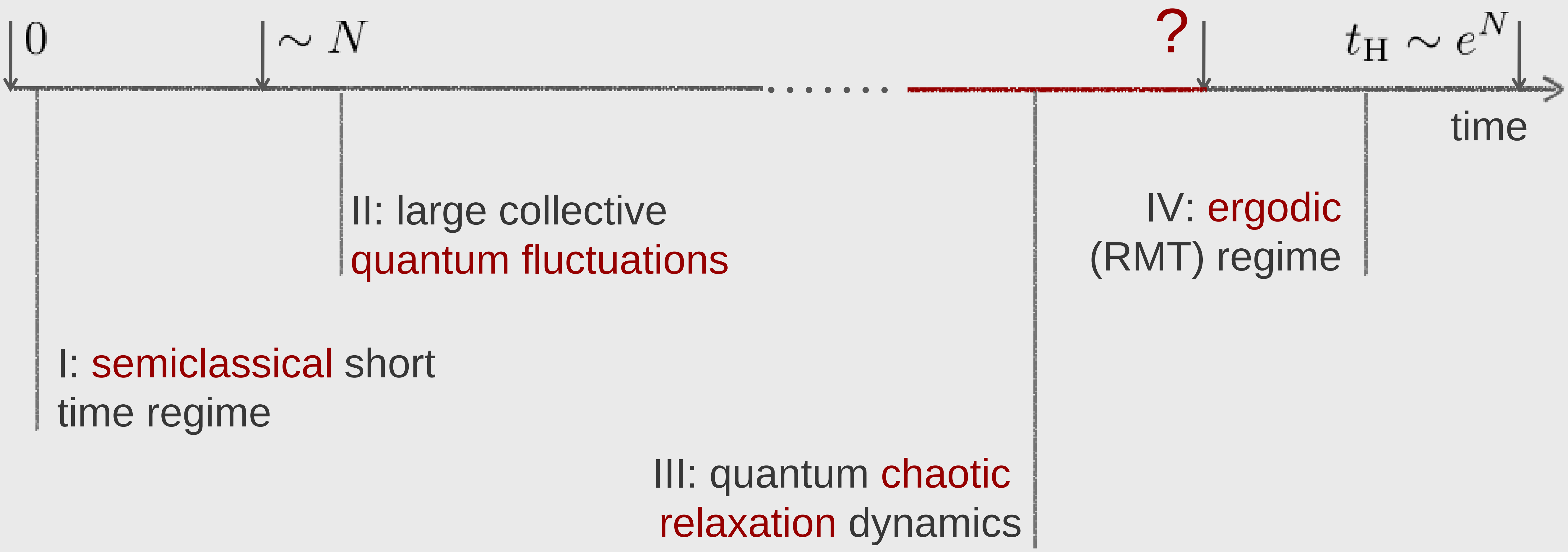
high energy scale

Three perspectives:

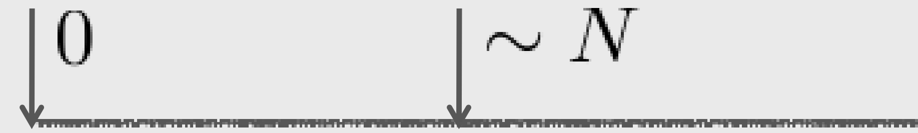
- random matrix theory
- strong correlation physics
- holography

stages of SYK dynamics

SYK timeline



I: semiclassical short time regime



Chaos manifests itself in exponential decay of dynamical correlation functions.

Prominent example: the out-of-time ordered (OTO) correlation function

(Larkin & Ovchinikov, 69)

$$F(t) = \mathrm{tr}\left(e^{-\beta \hat{H}} \hat{X} \hat{Y}(t) \hat{X} \hat{Y}(t)\right)$$

in its SYK specific incarnation

$$F(t) = \mathrm{tr}\left(e^{-\frac{\beta \hat{H}}{4}} \hat{X} e^{-\frac{\beta \hat{H}}{4}} \hat{Y}(t) e^{-\frac{\beta \hat{H}}{4}} \hat{X} e^{-\frac{\beta \hat{H}}{4}} \hat{Y}(t)\right)$$

Short time dynamics defined by chaos bound $T/\hbar$ (Maldacena & Stanford, 16).

$$F(t) = 1 - \frac{\beta e^{2\pi t/\beta}}{64\pi M} + \mathcal{O}(e^{\pi t/\beta}/M)$$

II: Collective fluctuation regime

$\sim N$

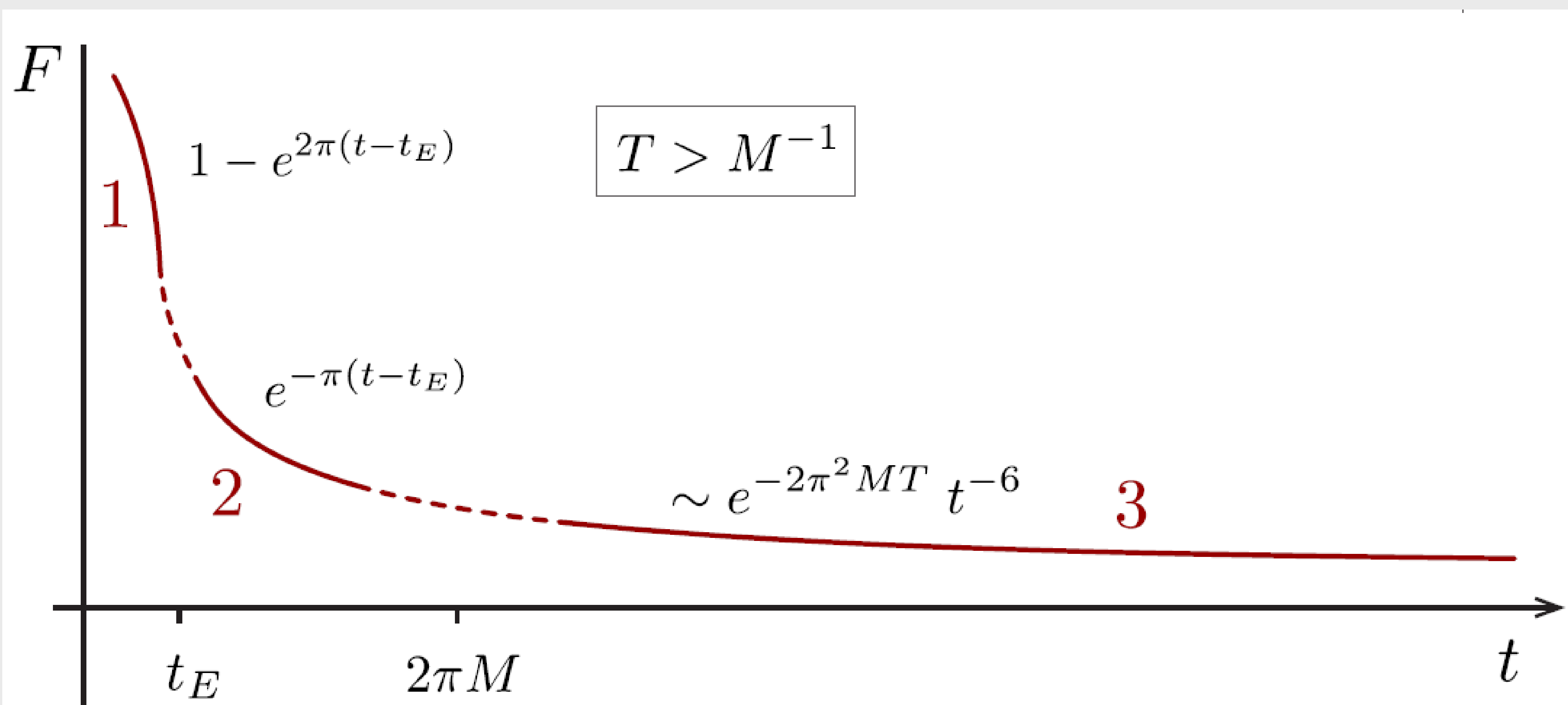
SYK model supports spontaneously broken infinite dimensional conformal symmetry.

Strong quantum fluctuations of Goldstone modes at time scales larger than
(DB, Altland and Kamenev, 17)

numerical factor

$$M = \frac{b^2}{32J} N \log(N)$$

OTO time profile turns into a universal $\sim t^{-6}$ power law

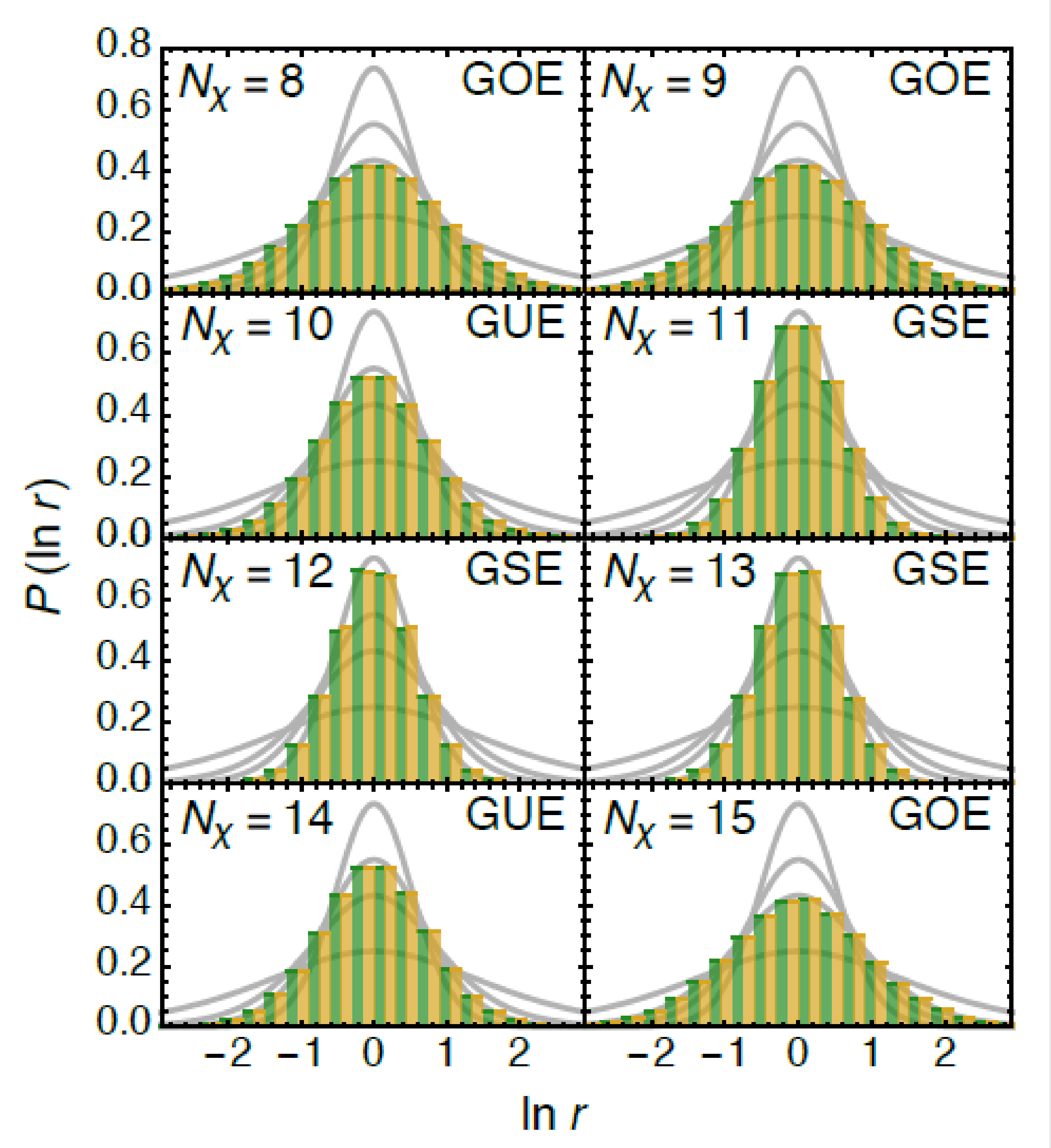


IV: Ergodic long time regime

Ergodic long time dynamics diagnosed in universal correlations of many body spectra

(i) Level spacing distribution:

$$r_n = \frac{E_n - E_{n+1}}{E_{n+1} - E_{n+2}}$$



Yi-Zhuang You, Andreas Ludwig, Cenke Xu, 16

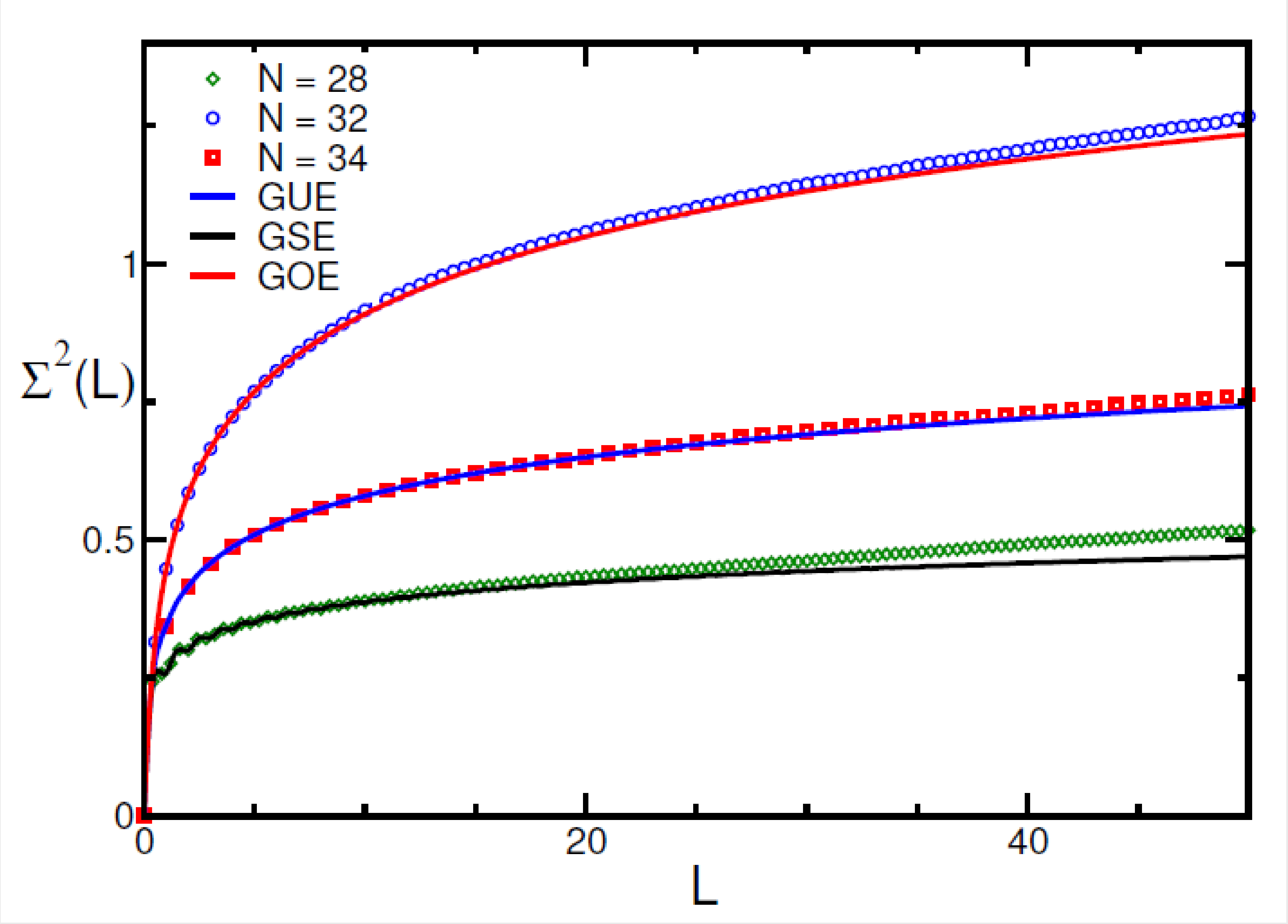
Note: depending on the value of *N* mod 8 the model realizes different symmetries

IV: Ergodic long time regime

Ergodic long time dynamics diagnosed in universal correlations of many body spectra

(ii) Level number fluctuations:

$$\Sigma^2(L) = \Delta^{-2} \left\langle \left[\int_{E-L/2}^{E+L/2} \rho(E')dE' \right]^2 \right\rangle_c$$



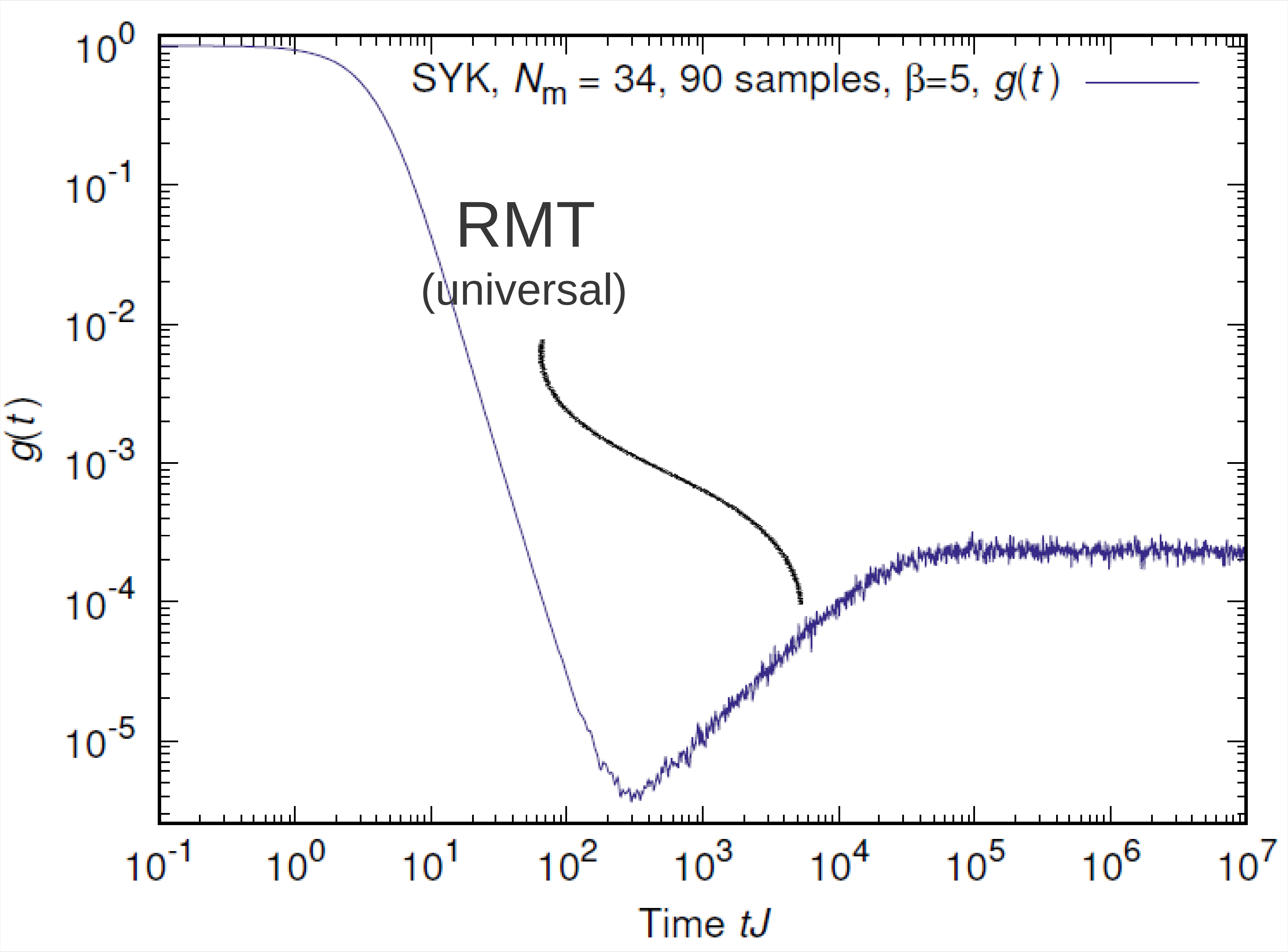
Verbaarschot, Garcia-Garcia, 16

IV: Ergodic long time regime

Ergodic long time dynamics diagnosed in universal correlations of many body spectra

(iii) Spectral form factor:

$$g(t) = \frac{\left\langle \left| \mathrm{tr}(e^{-(\beta+it)H}) \right|^2 \right\rangle}{\left\langle Z(\beta) \right\rangle^2}$$



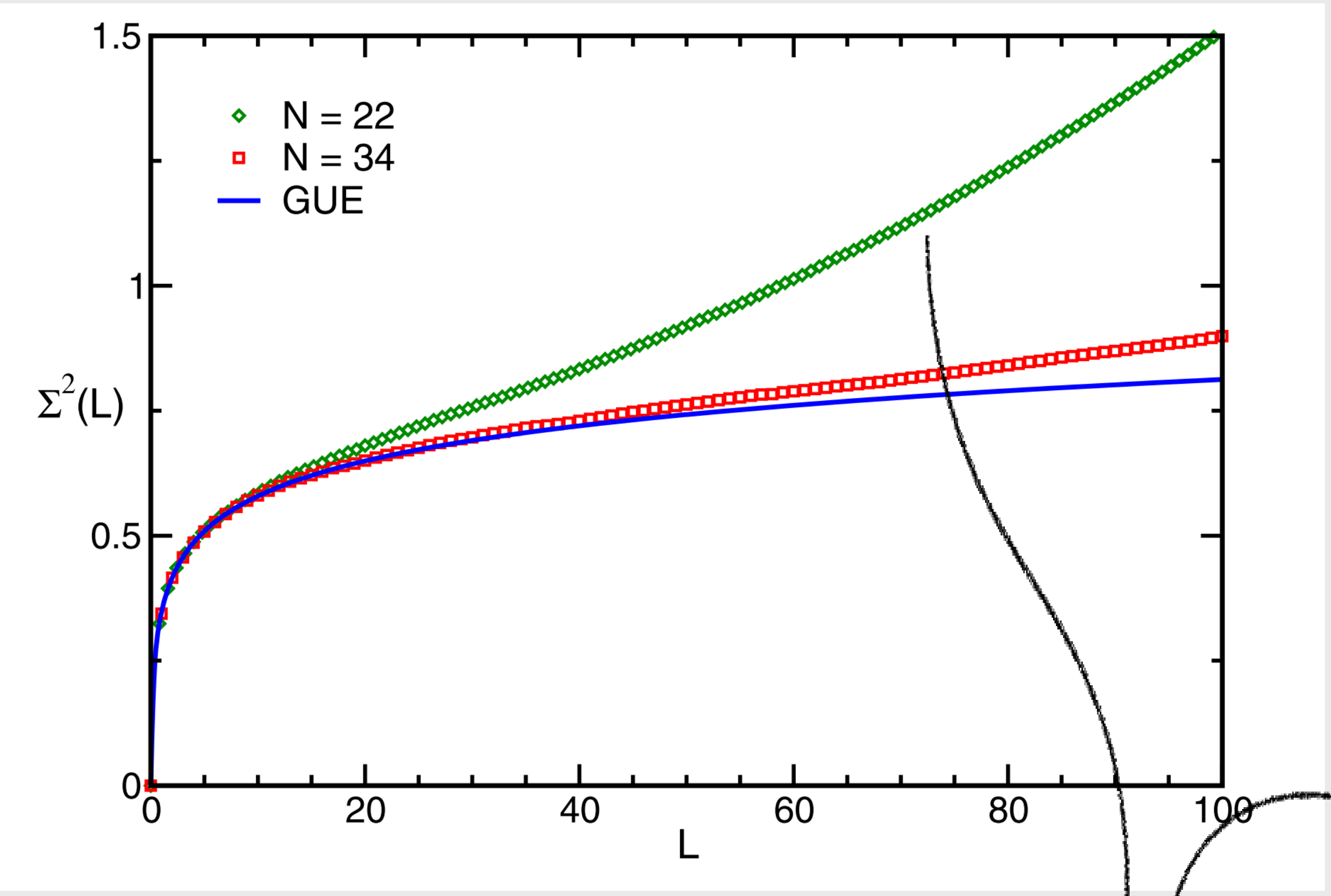
Cotler et al., 16

III: Pre ergodic regime

? $t_{\mathrm{H}} \sim e^{N}$

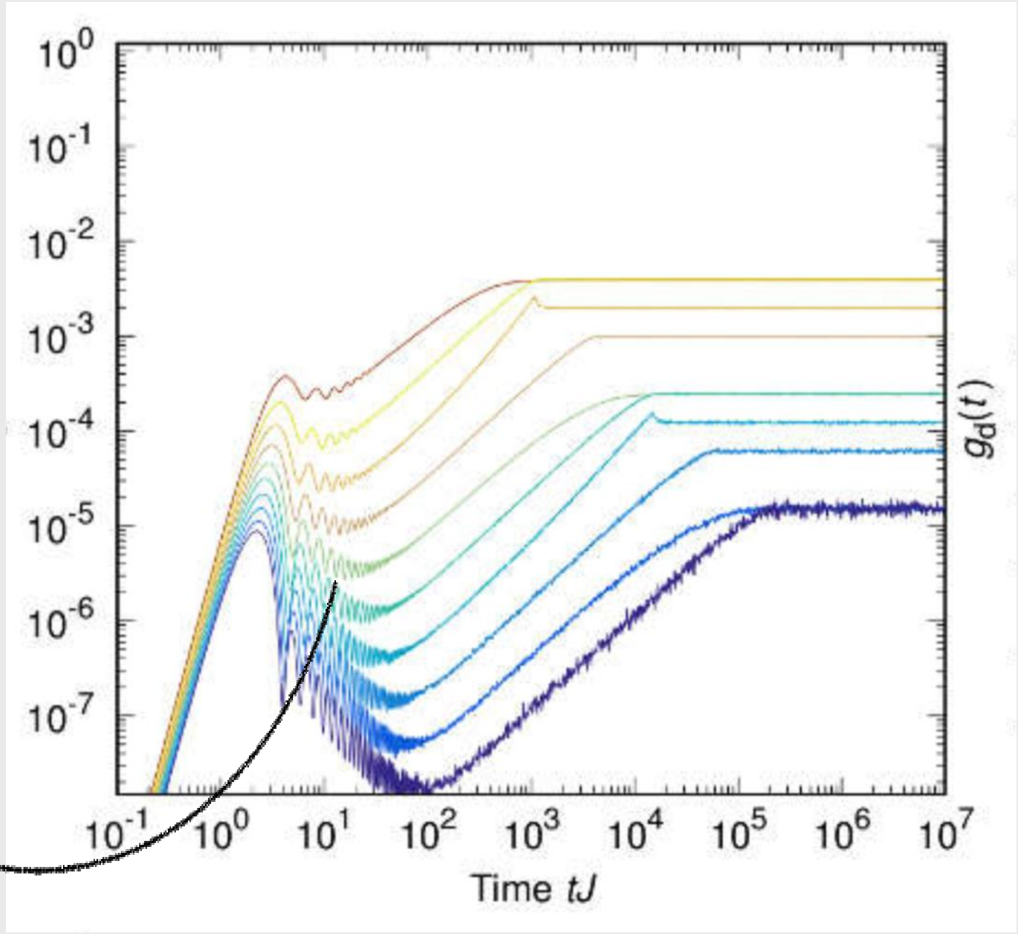
For times shorter than an ergodic time t_{erg} universal deviations from RMT behavior are observed.

Level number fluctuations



Verbaarschot, Garcia-Garcia, 16

Connected spectral form factor



Cotler et al., 16

not RMT

(still universal)

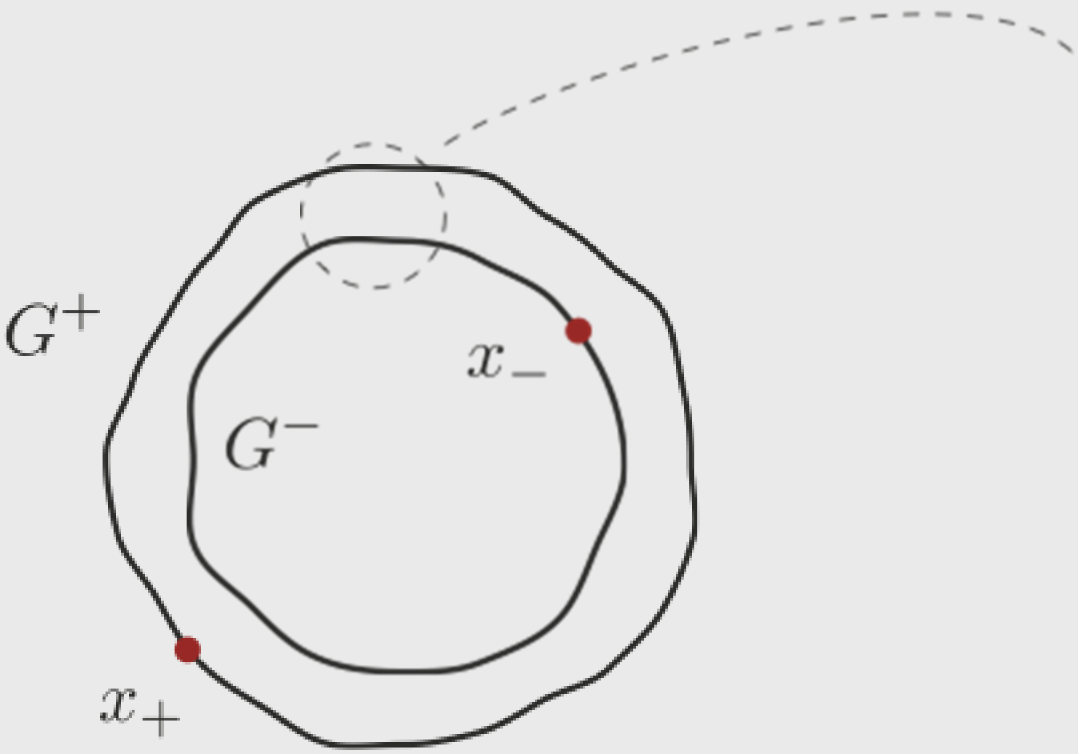
$$\boxed{R_2(\omega) \equiv \Delta^2 \left\langle \rho(E+\frac{\omega}{2})\rho(E-\frac{\omega}{2}) \right\rangle_c}$$

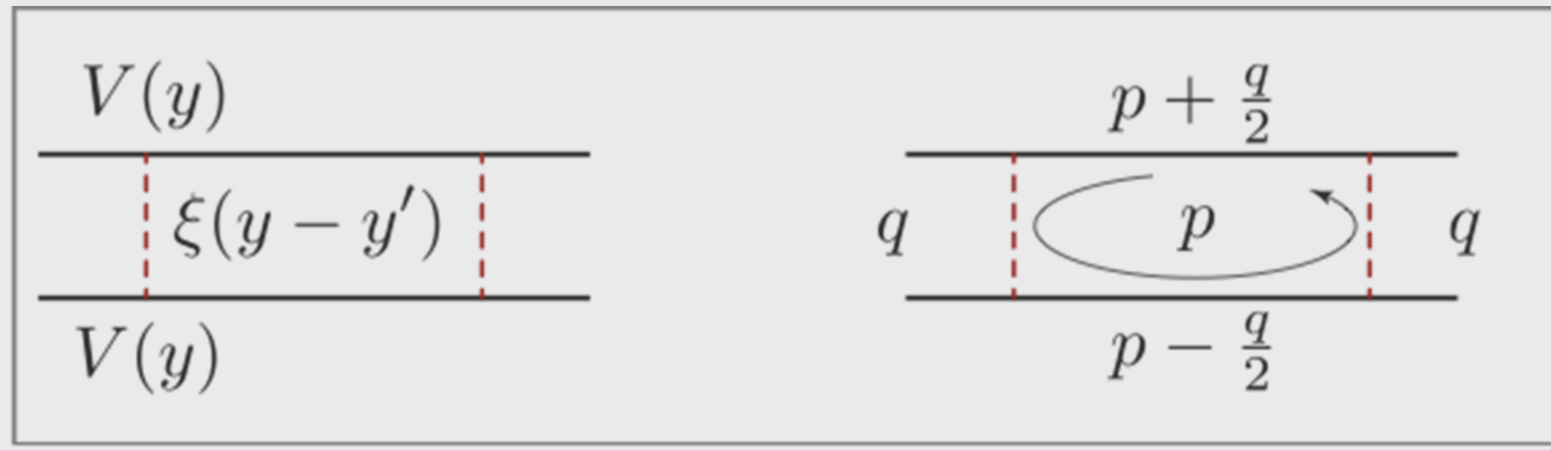
Spectral correlations in mesoscopics

Compare to the physics of dirty metals

Hamiltonian: $\hat{H} = -\frac{1}{2}\Delta_x + V(x), \quad \langle V(x)V(y)\rangle \propto \delta(x-y)$

white noise disorder

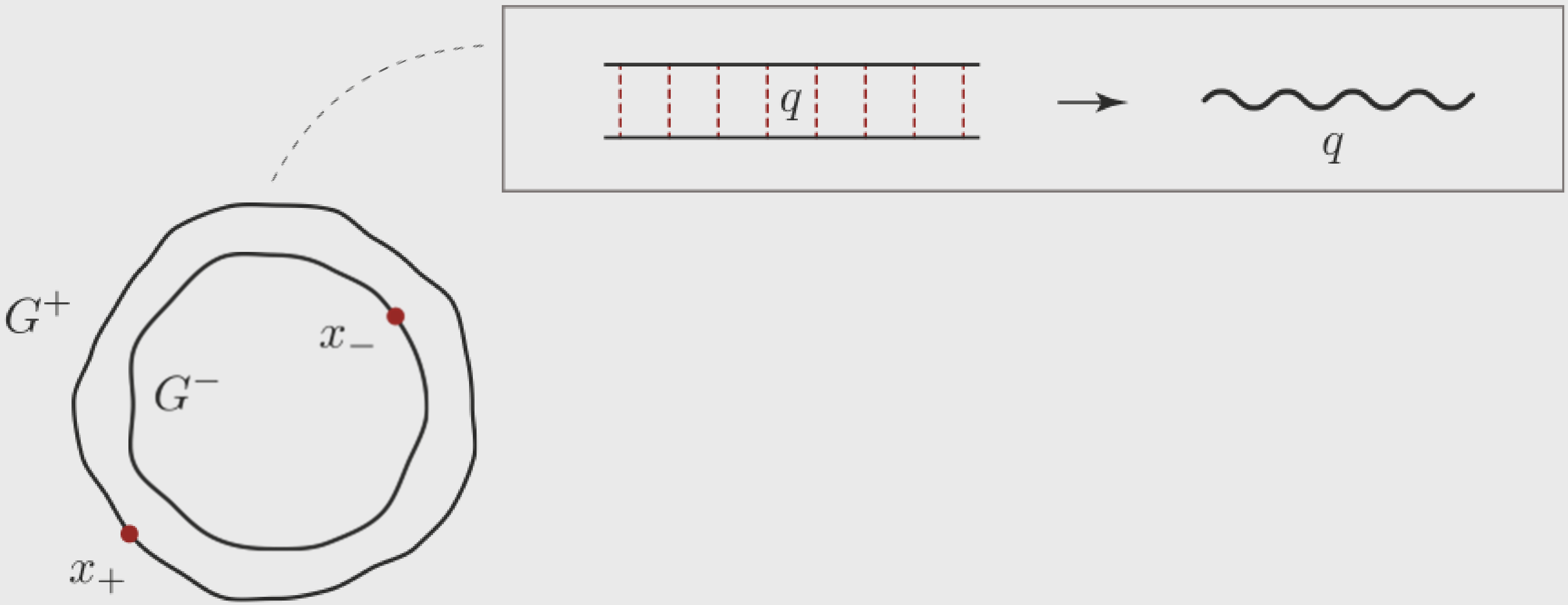




Momentum *q* difference conserves in scattering.

Interested in two-point function: $C(\omega) = \left\langle \mathrm{tr}\left(G^+\left(\varepsilon + \frac{\omega}{2}\right)\right)\mathrm{tr}\left(G^-\left(\varepsilon - \frac{\omega}{2}\right)\right)\right\rangle_c$

dirty metals cont'd



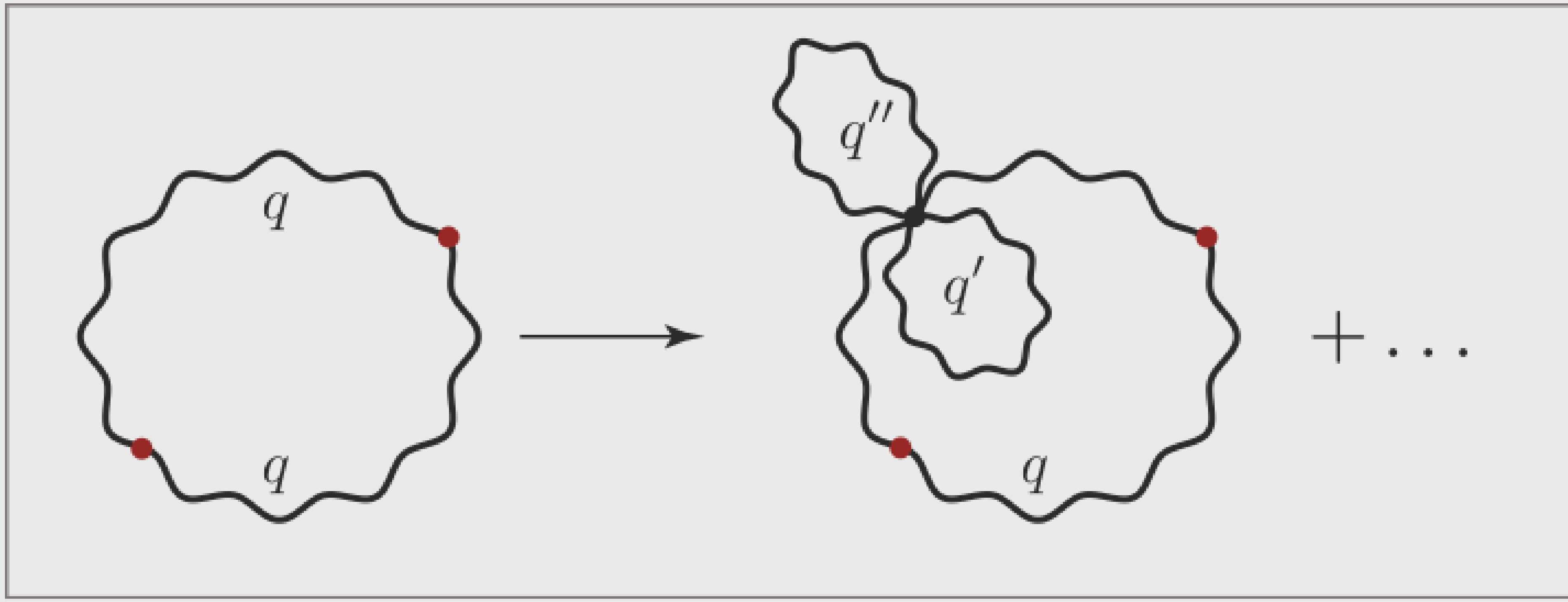
Modes characterized by

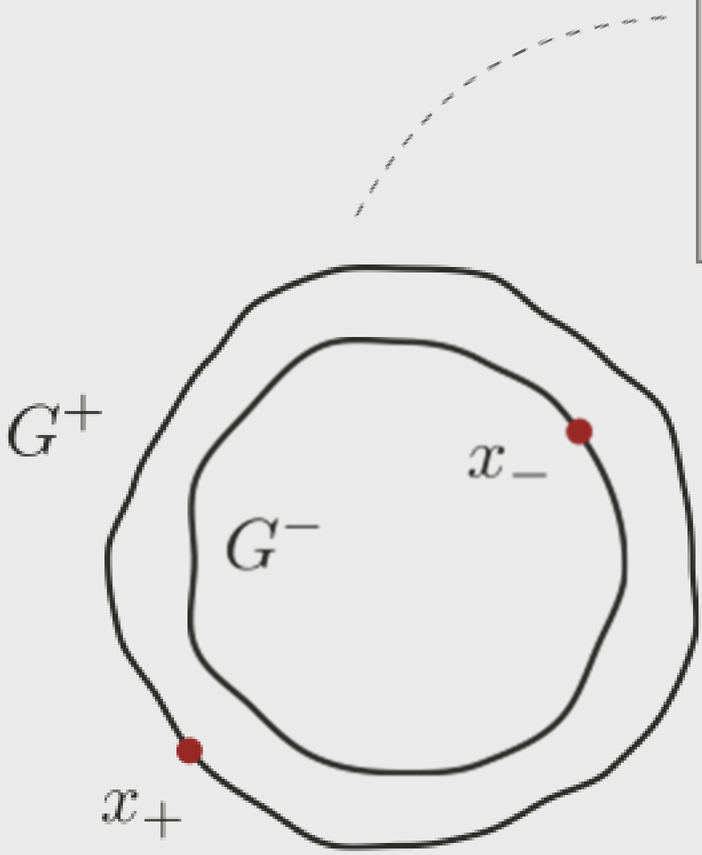
(i) 'discrete quantum numbers' $q \in \frac{2\pi}{L}\mathbb{Z}^3$

(ii) propagator, $\left(Dq^2 + i\omega\right)^{-1}$

(iii) physical interpretation as irreversible relaxation modes

Dirty metals cont'd





$$R_2(\omega) = R_{2,\mathrm{RMT}}(\omega) + \frac{1}{2}\left(\frac{\Delta}{\pi}\right)^2 \mathrm{Re}\sum_{q\neq 0}\frac{1}{(i\omega - Dq^2)^2}$$

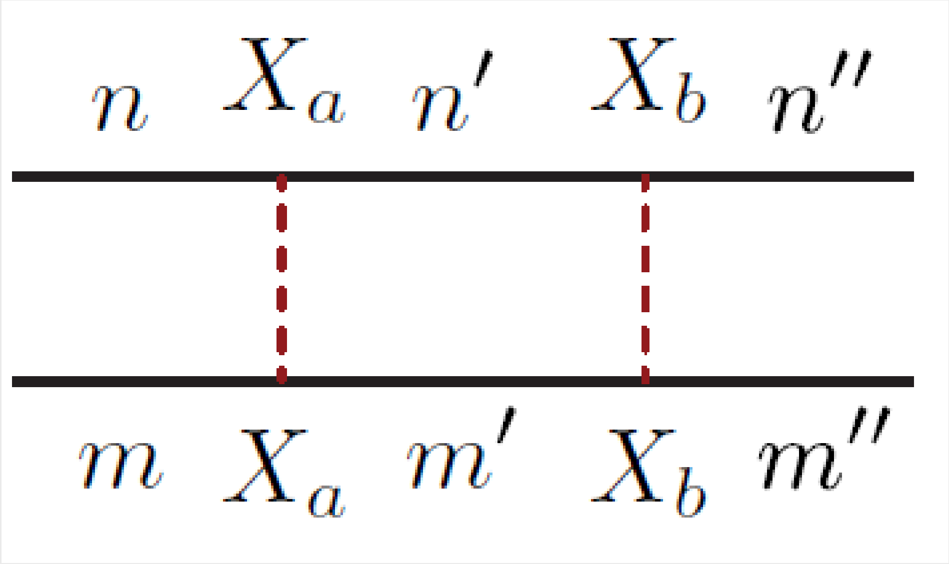
A. Andreev, B. Altshuler, 95

GUE spectral two-point function:

$$R_{2,\mathrm{RMT}}(\omega) = \Delta\delta(\omega) - \left(\frac{\sin(\pi\omega/\Delta)}{\pi\omega/\Delta}\right)^2$$

relaxation dynamics in Fock space

Majorana relaxation modes



$$X_a = \chi_{i_1}\chi_{i_2}\chi_{i_3}\chi_{i_4}$$

Scattering of interaction vertices, $X_a, X_b,$ etc.

$|m\rangle = (0,1,1,0,\ldots) \in V$ - many-body states in Fock space, $\dim(V) = 2^{N/2}$

Summing up ladder diagrams

$$\Pi^{n'm'} = \sum_{nm} \langle n'|X_a|n\rangle\langle m|X_a|m'\rangle \Pi^{nm}$$

composite mode (matrix in Hilbert space)

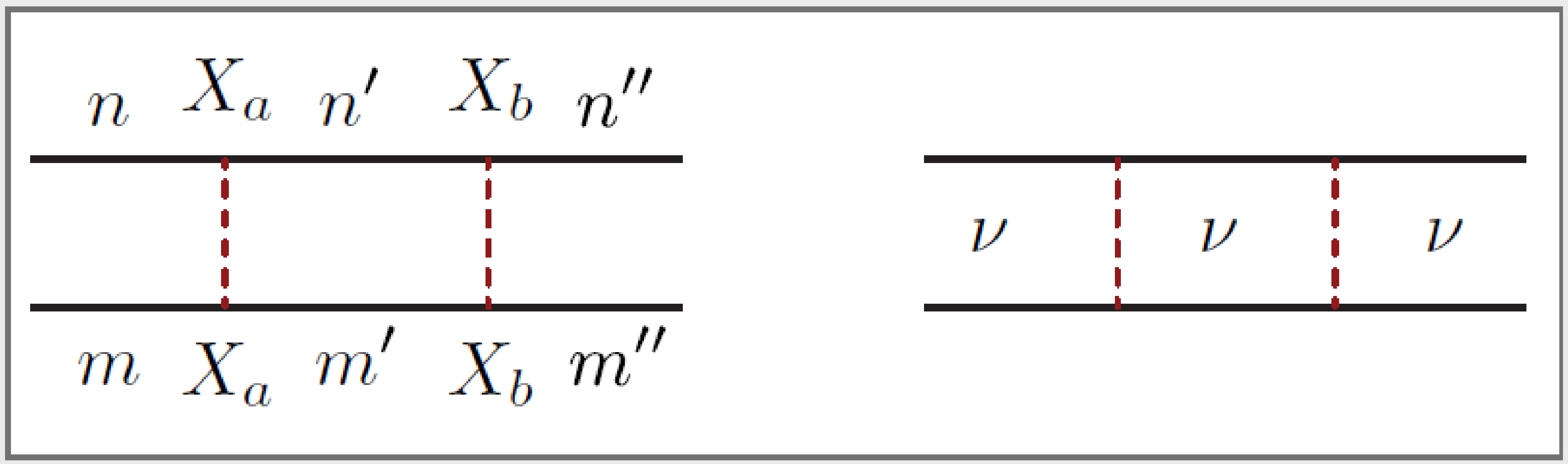
$$\Pi' = X_a \Pi X_a$$

Fourier transform in Hilbert space of matrices

$$\dim(V) = 2^{N/2}$$ - dimension of many-body Fock space

$$X_\mu = \chi_{\mu_1}\chi_{\mu_2}\ldots\chi_{\mu_l}$$ - full basis set in DxD dimensional space $V \otimes V^*$, i.e. the space of operators acting in Fock space

$$\mathrm{tr}\left(X_\mu X_\nu^+\right) = D\delta_{\mu\nu}$$ - basis is orthogonal



Scattering conserves μ–state

‘Fourier transform‘

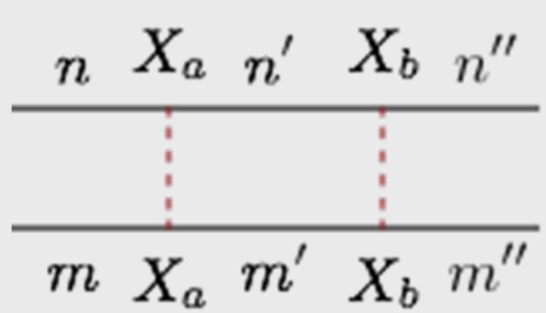
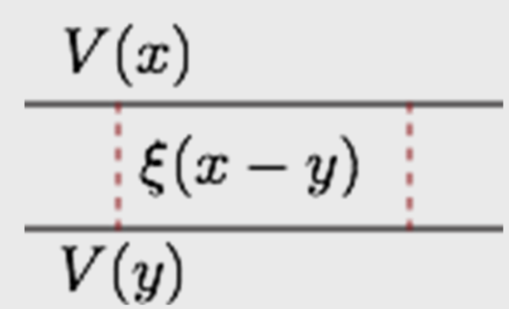
$$\Pi = \frac{1}{\sqrt{D}}\sum_\nu \pi_\nu X_\nu$$

Inverse ‘Fourier transform‘

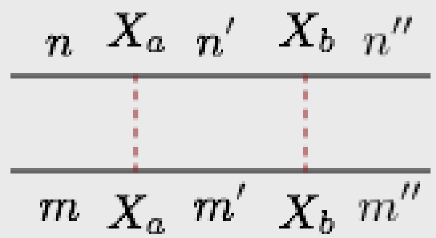
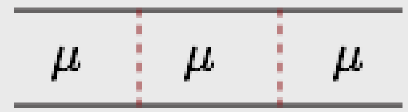
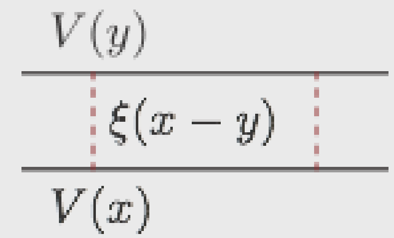
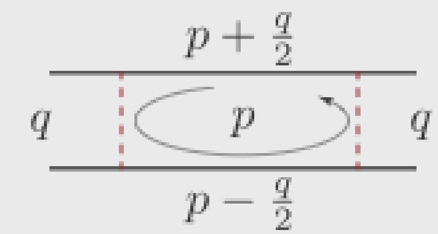
$$\pi_\nu = \frac{1}{\sqrt{D}}\mathrm{tr}\left(\Pi X_\nu^+\right)$$

SYK model vs. dirty metal (summary)

$$\hat{H} = \sum_{ijkl}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l \equiv \sum_a J_a \hat{X}_a, \qquad a = (i,j,k,l)$$

	SYK	dirty metal
Hilbert space	Fock space of *N/2* fermions	function space
dimension	$2^{N/2}$	∞
basis states	$m = (1,0,0,1,\dots)$	$\lvert x\rangle$
scattering vertex		
scattering states	$\lvert n\rangle \otimes \langle m\rvert$	$\lvert x\rangle \otimes \langle y\rvert$
basis of conserved states	$\hat{X}_\mu \equiv \chi_{\mu_1}\chi_{\mu_2}\cdots\chi_{\mu_k},$ $\mu \equiv (\mu_1,\mu_2,\dots,\mu_k)$	$\lvert p+\frac{q}{2}\rangle \otimes \langle p-\frac{q}{2}\rvert$

Spectrum of relaxation modes

	SYK	dirty metal
conserved modes	 	 
eigenvalues	$\epsilon(\lvert\mu\rvert) - i\omega$	$Dq^2 - i\omega$

# of Majoranas in state

$$\varepsilon(k) \sim 2^{N/2}\Delta \times k/N, \quad k << N$$

$$\varepsilon(k) \sim 2^{N/2} N^2 \Delta, \quad k \sim N$$

$$\Delta \sim \frac{JN^{1/2}}{2^{N/2}}$$

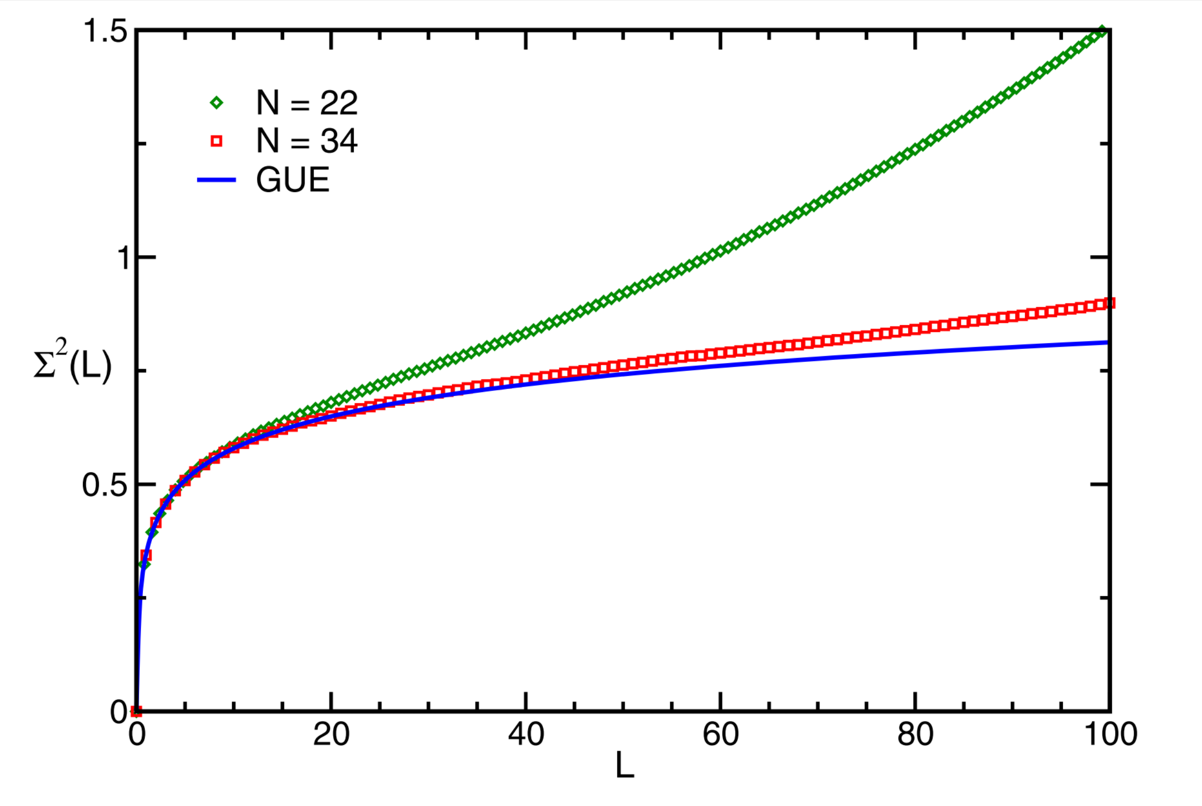
many body level spacing

SYK spectral correlation function

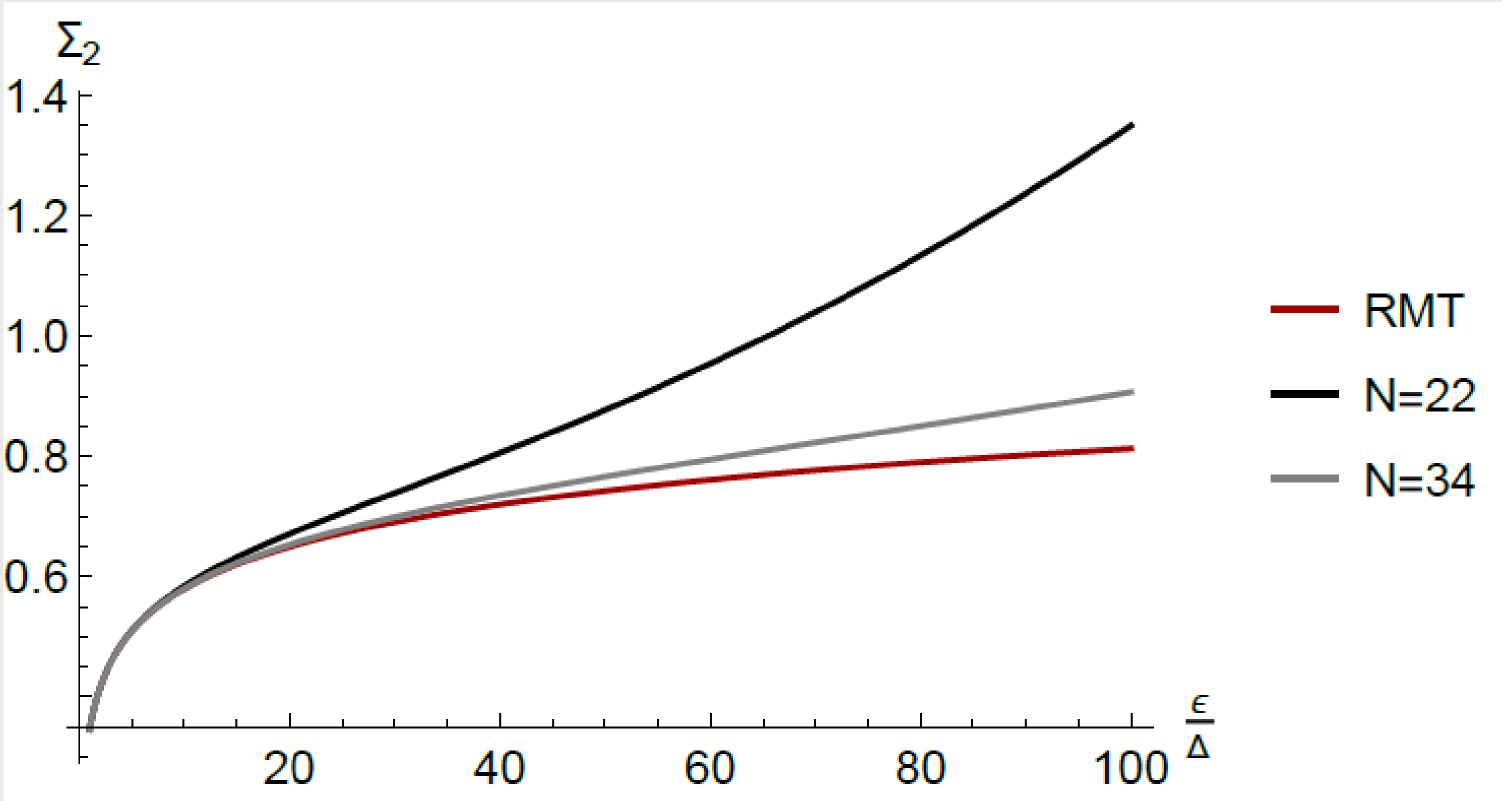
$$R_2(\omega) = R_{2,\mathrm{RMT}}(\omega) + \frac{1}{2}\left(\frac{\Delta}{\pi}\right)^2 \mathrm{Re} \sum_{k\neq 0,\mathrm{even}} \binom{N}{k} \frac{1}{(i\omega - \epsilon(k))^2}$$

comparison to numerical data I: number variance

Verbaarschot, Garcia-Garcia, 16



Altland, DB, 18



Ergodic energy scale is

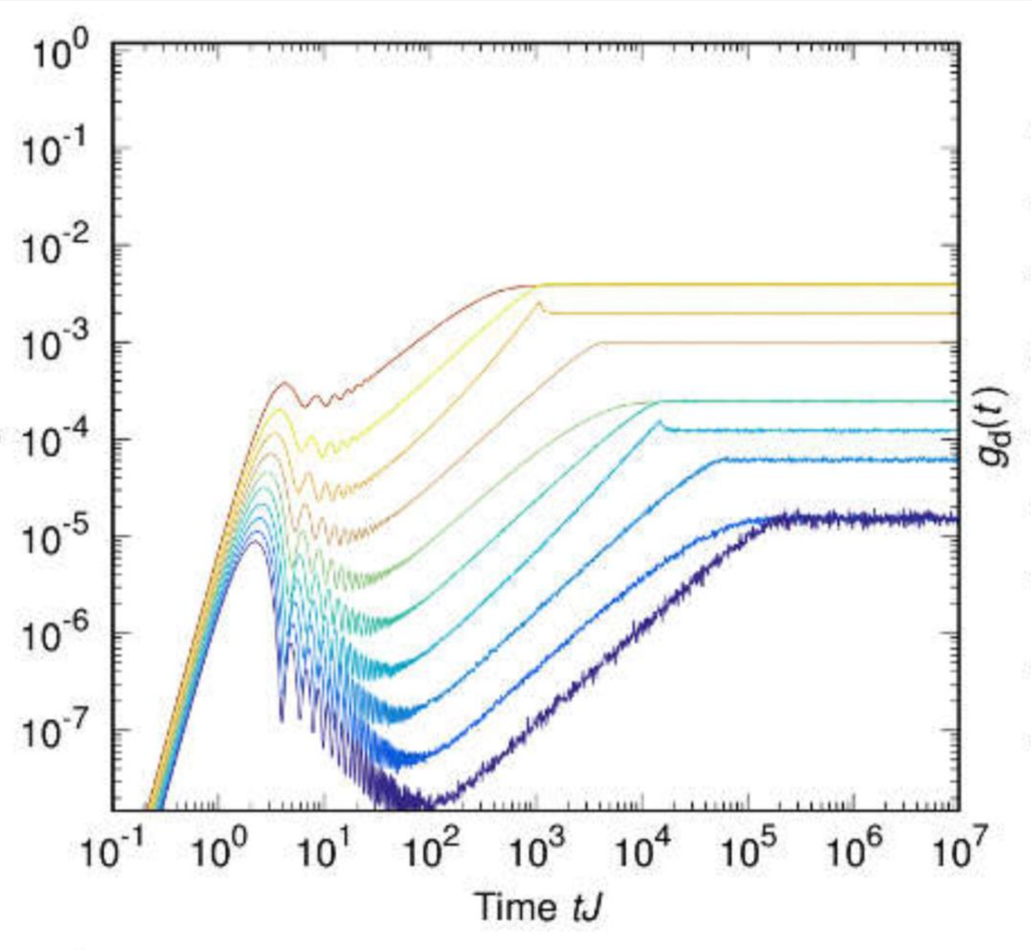
$$\omega_{\mathrm{erg}} \sim \Delta N^2$$

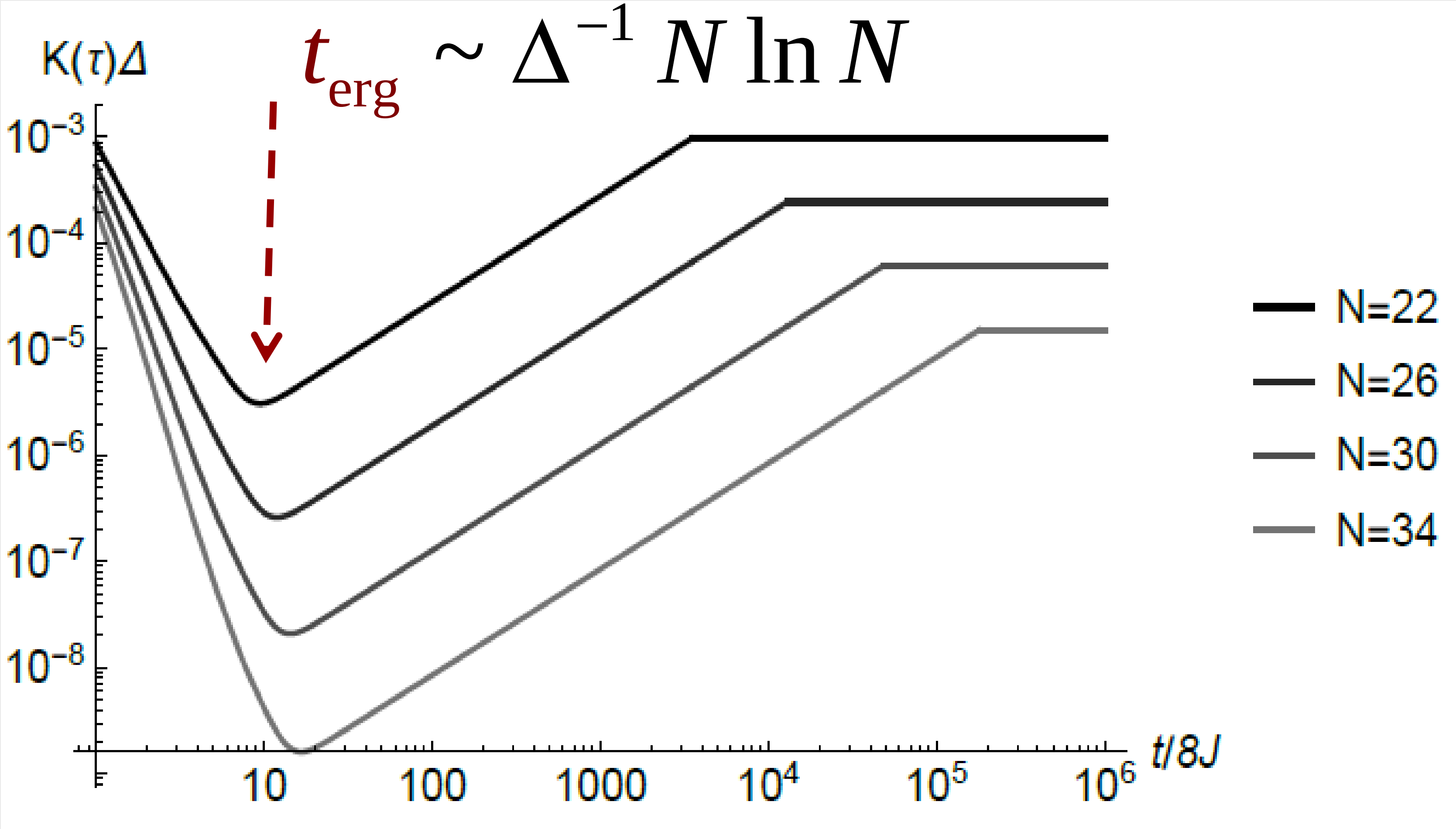
$$\Sigma_2(\varepsilon) = \frac{1}{\pi}\ln(\varepsilon/\Delta) + \frac{4\pi\varepsilon^2}{\Delta^2 N^4}$$

SYK spectral correlation function

comparison to numerical data II: spectral form factor

$$K(\tau) = K_{\mathrm{RMT}}(\tau) + \tau \sum_{k\neq 0,\, k\in \mathrm{even}} \binom{N}{k} e^{-\tau \frac{2\pi\varepsilon(k)}{\Delta}}$$





Cotler et al. 17

Altland, DB, 18

all in all: very good parameter free agreement with numerical data.

SUSY field theory (nonlinear σ-model) in mesoscopics (Efetov '83)

$$S[Q]=\frac{gL^{2-d}}{8}\int d^dx\,\mathrm{str}\left(\nabla Q\right)^2+\frac{i\pi L^{-d}}{4\Delta}\int d^dx\,\mathrm{str}\left(Q\omega\Lambda\right)$$

bare conductance

single-particle level spacing

Field Q(x) is the supermatrix in $\mathrm{bf}\otimes RA$ space

$Q\in U(2|2)/U(1|1)\otimes U(1|1)$ - coset space (in class A)

Here $\Lambda=1^{\mathrm{bf}}\otimes\tau_3^{\mathrm{RA}}$ is the so-called trivial saddle point

Universal RMT limit (d=0)

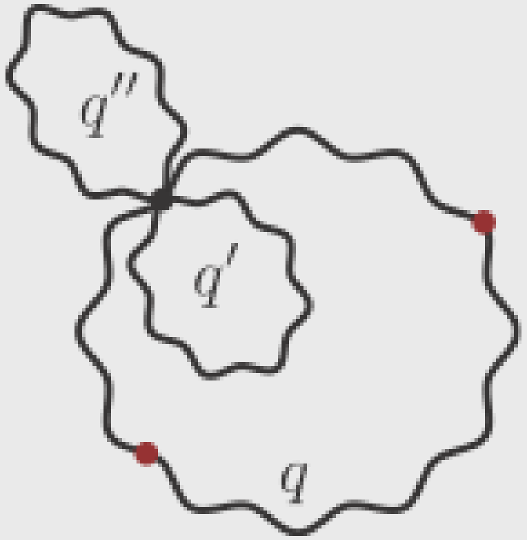
The model becomes semiclassically exact:

$$R_{2,\mathrm{RMT}}\left(\omega\right)=-\tfrac{1}{2}\left(\tfrac{\Delta}{\pi}\right)^2\mathrm{Re}\frac{1}{\left(\omega+i0\right)^2}\left(1-e^{-S_0[\bar{\Lambda}]}\right)$$

non-trivial saddle point (Andreev, Altshuler): $\bar{\Lambda}=(P^{\mathrm{bb}}-P^{\mathrm{ff}})\otimes\tau_3^{\mathrm{RA}}$
breaks SUSY

Appendix

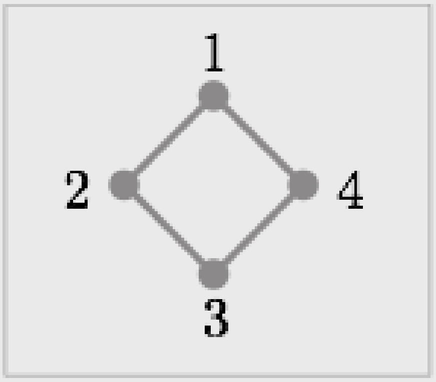
coupling between Fock space relaxation modes? (The analog of dirty metal weak or strong localization physics?)

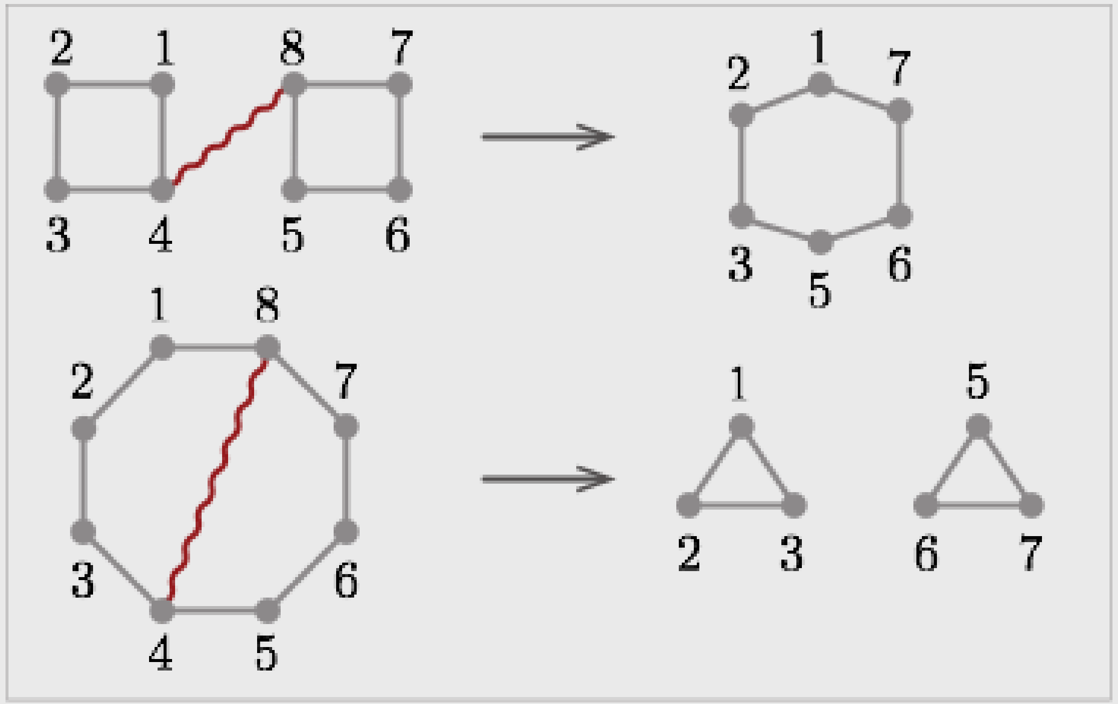


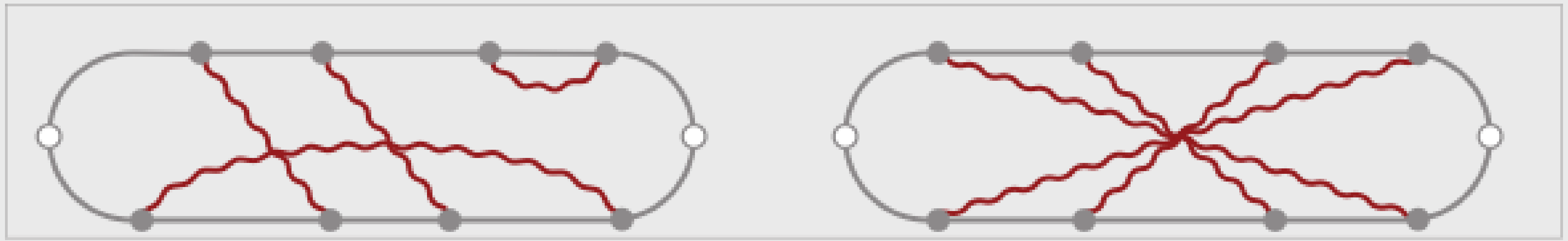
Possible expectations: No: no infrared singularities like low-dimensional metals. Yes: there are exponentially many modes.

Need to take detailed look. Short answer: nonlinear couplings of *k* modes exponentially suppressed in

$$\sim \exp\left(-k \ln N\right)$$







summary

identified a high density set of conserved modes
in an interacting (Majorana) fermion system

dominate fluctuations beyond a non-universal
"Thouless energy"

a case study of chaotic relaxation in a
nonlinear many body environment